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Separation of variables in Einstein spaces. I. Two ignorable and one null coordinate

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Abstract. We compute and classify all Einstein spaces and their corresponding coordinate systems such that the Hamilton–Jacobi equation admits a separation of variables with two ignorable and one essential null coordinates. We find that all such spaces correspond to plane-fronted gravitational waves of Petrov type N. We also give necessary conditions which a general space–time must have in order to admit coordinates of the type studied. Furthermore, the integrability conditions for Helmholtz (Schrödinger) separability are completely solved.

1. Introduction

This is the first of a series of papers devoted to the classification of separable coordinate systems for four-dimensional Einstein spaces and the classification of the corresponding Einstein spaces. We shall pay particular attention to Einstein spaces with the Lorentzian signature $(+++-)$ used in general relativity. The utility in general relativity of finding coordinates which separate the Hamilton–Jacobi equation is two fold. First, one can possibly find new space–times which, for example, satisfy the vacuum Einstein equations; second, even if the space–time is known the complete integrals of the Hamilton–Jacobi equation allow one to integrate the geodesic equations and thus study global properties of the spaces. Both of these possibilities were first realised by Carter (1968a, b). Since that time there has been much work on finding separable coordinates in general relativity, but any kind of classification has been lacking. We refer the reader to the recent review article of Benenti and Francaviglia (1980) for a comprehensive discussion of separability as applied to general relativity along with a fairly comprehensive bibliography. For the treatment of separability of the partial differential equations of mathematical physics and its connection with Lie theory the reader is referred to the book of Miller (1977).

In Boyer *et al* (1978) the authors gave a classification of canonical forms for the separability of the Hamilton–Jacobi equation in four-dimensional complex Riemannian spaces based on the number of ignorable and essential null coordinates. We also gave necessary and sufficient conditions for the separation of the corresponding Helmholtz (or Schrödinger) equation. In this paper we concentrate on type D separable coordinates in Boyer *et al* (1978), that is, those with two ignorable and one essential null coordinates. These coordinates arise in space–times describing gravitational waves. In fact, we shall show that the only vacuum Einstein spaces which admit such coordinates are the plane-fronted parallel waves (pp waves) (Pirani 1957, Bondi *et al* 1959, Plebański 1979, Kundt 1961). Some partial results concerning separation in such space–times have been obtained previously (Matravers 1976, Collinson and Fugere 1977).

Another interesting mathematical problem which arises is that of a local obstruction theory. The simplest example of this is the celebrated fundamental theorem: A pseudo-Riemannian manifold admits a Cartesian coordinate chart if and only if its curvature tensor vanishes. Here we prove that necessary conditions that a four-dimensional pseudo-Riemannian manifold admits a coordinate chart of type D are that it has a geodesic principal null direction and a complementary foliation of codimension one with totally geodesic leaves. In terms of the physical optical parameters, this means that the principal null direction is geodesic shearfree, twistfree and non-expanding. These necessary conditions can be formulated in terms of the vanishing of certain Spencer cohomology groups, but a clear picture of this obstruction theory in a general setting has yet to emerge.

Finally we mention that ours is the first work where a complete classification of Einstein spaces admitting a specific type of separation of variables is given. For more details of the computations involved the reader is referred to Boyer *et al* (1979).

2. Separation of variables

In Boyer *et al* (1978) the authors gave a classification of separable coordinate systems for the Hamilton–Jacobi equation

$$g^{\mu\nu}(x)S_{x^\mu}S_{x^\nu}=E \tag{2.1}$$

in complex Riemannian spaces. We also gave necessary and sufficient conditions which enable separation of the Helmholtz equation

$$\Delta_4\Phi=E\Phi \tag{2.2a}$$

where Δ_4 is the Laplace–Beltrami operator

$$\Delta_4=\sum_{i,j=1}^{4}|g|^{-1/2}\partial_{x^\mu}(|g|^{1/2}g^{\mu\nu}\partial_{x^\nu}). \tag{2.2b}$$

Our classification is based on the number of ignorable coordinates. If $\{x\}$ is a separable coordinate system with x^i an essential (non-ignorable) coordinate we distinguish two types: An essential variable x^i is said to be of type 1 if the separated ordinary differential equation is linear in S_{x^i} and type 2 if it is quadratic in S_{x^i}. We call x^i a *null coordinate* if $g(dx^i, dx^i)=0$. Then it is easy to see that an essential coordinate x^i is null if and only if it is of type 1.

Of the four-dimensional complex Riemannian spaces admitting a coordinate system with (2.1) separable, there are three with two ignorable coordinates—types C, D and E in Boyer *et al* (1978) having, respectively, 0, 1 and 2 essential null coordinates. Thus type E having two null coordinates is only possible for real spaces with signature $(++--)$. Type C for real spaces with Lorentzian signature $(+++-)$ includes the class investigated by Carter (1968b).

In this article we shall study type D coordinates in detail; in particular, we classify all Einstein spaces with Lorentzian signature. We shall treat the more general R-separation in forthcoming publications.

Actually as discussed previously (Boyer *et al* 1978, Kalnins and Miller 1979) it is not *a* separable coordinate system which interests us but an equivalence class of separable coordinates. We say that two separable coordinate systems $\{x^i\}$ and $\{x'^i\}$ are equivalent if they are related by the pseudogroup P of coordinate transformations defined by

$$x'^i = X^i(x^i) \qquad x'^\alpha = \sum_\beta A^\alpha_\beta x^\beta + \sum_i f^\alpha_i(x^i) \tag{2.3}$$

where x^i are essential and x^α (x^β) are ignorable coordinates, X^i, f^α_i are arbitrary analytic functions of the variable x^i, and A^α_β is a matrix of real (complex) numbers with $\det A^\alpha_\beta \neq 0$, i.e. $A \in \mathrm{GL}(k)$ where k is the number of ignorable coordinates.

One problem that arises is that a separable coordinate system with k ignorable coordinates can be equivalent to a separable system with greater than k ignorable coordinates (Benenti and Francaviglia 1980). We shall always consider the maximal number of ignorable coordinates as those which characterise the system. An explicit example of this phenomenon will be given shortly. Often by abuse of language, we shall refer to an equivalence class of separable coordinate systems as a separable coordinate system.

We now specialise to the case of four real (or complex) dimensions. In particular we are interested in the real case with the Lorentzian signature $(+++-)$. In our classification (Boyer *et al* 1978) in four dimensions the separable systems with two ignorable and one essential null coordinates fell into two subcases with the following contravariant metrics

$$(g^{\mu\nu}) = \frac{1}{K_1 - K_2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & a_2 & b_2 \\ 0 & a_2 & d_1 & f_1+f_2 \\ 0 & b_2 & f_1+f_2 & e_1 \end{pmatrix} \tag{D1}$$

$$(g^{\mu\nu}) = \frac{1}{K_1 - K_2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & d_1+d_2 & f_1 \\ 0 & 1 & f_1 & e_1 \end{pmatrix} \tag{D2}$$

where the subscripts $i = 1, 2$ indicate an arbitrary function depending only on the variable x^i.

However, *Lemma 1.* Every D2 coordinate system is equivalent to a D1 coordinate system.

We thus need only consider metrics of type D1 which we shall refer to as D. It is convenient at this point to introduce some new notation:

$$\begin{aligned} &h_\alpha = (a_2e_1 - b_2(f_1+f_2), b_2d_1 - a_2(f_1+f_2)) \\ &h_{\alpha\beta} = \begin{pmatrix} b_2^2 & -a_2b_2 \\ -a_2b_2 & a_2^2 \end{pmatrix} \qquad h = (f_1+f_2)^2 - d_1e_1 \\ &A = (K_1 - K_2)^{-1/2} \qquad \phi = (a_2^2e_1 + b_2^2d_1 - 2a_2b_2)^{-1/2}. \end{aligned} \tag{2.4}$$

Then the covariant metric takes the form

$$ds^2 = A^{-2}[(dx^1)^2 + \phi^2 h (dx^2)^2 + 2\phi^2 h_\alpha \, dx^2 \, dx^\alpha + \phi^2 h_{\alpha\beta} \, dx^\alpha \, dx^\beta]. \quad (2.5)$$

In order to have the correct signature $(+++-)$ we must require that $K_1 - K_2 > 0$ and $a_2^2 e_1 + b_2^2 d_1 - 2a_2 b_2 > 0$.

The Killing tensors which describe systems of type D are easily found to be

$$L_1 = \partial_{x^4} \qquad L_2 = \partial_{x^3}$$

$$L_3 = (K_1 - K_2)^{-1}[2K_1(a_2\partial_{x^2x^3} + b_2\partial_{x^2x^4} + f_2\partial_{x^3x^4}) + K_2(\partial_{x^1x^1} + d_1\partial_{x^3x^3} + 2f_1\partial_{x^3x^4} + e_1\partial_{x^4x^4})]. \quad (2.6)$$

Lemma 2. If b_2 (or equivalently a_2) vanishes, then a D metric is equivalent to a metric of the form

$$ds^2 = (K_1 - K_2)[(dx^1)^2 + (f_1^2/e_1 - d_1)(dx^2)^2 + 2\,dx^2\,dx^3 - 2(f_1/e_1)\,dx^2\,dx^4 + e_1^{-1}(dx^4)^2].$$

Thus if $K_2' = 0$, a D metric with $b_2 = 0$ is equivalent to a metric with three ignorable coordinates (class B in Boyer *et al* (1978)).

The purpose of this article is to prove the following.

Theorem 1. Let (M, g) be a non-flat Einstein space which admits a separable coordinate system $\{x^k\}$ with strictly two ignorable coordinates and one null essential coordinate (type D in Boyer *et al* (1978)). Then (M, g) with the coordinates $\{x^k\}$ is up to equivalence determined by one of the following metrics.

(i) $$ds^2 = A^{-2}[(dx^1)^2 + (x^1)^2(dx^4)^2 - (\alpha_0(x^1)^2 + \alpha_1 \ln x^1)(dx^2)^2 + 2\,dx^2\,dx^3 + 2\varepsilon\,dx^2\,dx^4].$$

(ii) $$ds^2 = A^{-2}[(dx^1)^2 + (dx^4)^2(x^1)^2(dx^2) + 2\,dx^2\,dx^3 + 2(\alpha_1 x^1 + \alpha_0)\,dx^2\,dx^4]$$

where

$$A = \begin{cases} \cos \lambda \omega^{1/2} x^2 & \omega > 0 \\ \cosh \lambda |\omega|^{1/2} x^2 & \omega < 0 \\ x^2 & \omega = 0 \end{cases}$$

$\omega = \varepsilon - \alpha_0$ for case (i) and $\omega = \frac{1}{4}\alpha_1^2 \pm \frac{1}{2}$ for case (ii), α_1, α_0 are arbitrary constants ($\alpha_1 \neq 0$ in (i)) and $\varepsilon = 0, 1$.

(iii) $$ds^2 = A^{-2}\{(dx^1)^2 + \phi^2(dx^4)^2 + 2\,dx^2\,dx^3 - [\alpha_1(x^1)^2 + \alpha_0 x^1 - \phi^2\tfrac{1}{4}(x^1)^2 - \beta^2/4\phi^2](dx^2)^2 - \phi^2 x^1\,dx^2\,dx^4\}$$

where ϕ is an arbitrary function of x^2 and A is determined by

$$2A'' + (\tfrac{3}{8}\phi^2 - \alpha_1 - \phi''/\phi)A = 0.$$

(iv) $$ds^2 = A^{-2}[(dx^1)^2 + \phi^2(dx^4)^2 - (\alpha_1(x^1)^2 - 1/4\phi^2)(dx^2)^2 + 2\,dx^2\,dx^3]$$

with ϕ again an arbitrary function of x^2 and

$$2A'' - (\alpha_1 + \phi''/\phi)A = 0.$$

(In (iii) and (iv) α_0, α_1 and β are arbitrary constants.)

Corollary. Let (M, g) be as in theorem 1; then:

(i) The space has Petrov type N.
(ii) The totally geodesic surfaces $x = \text{constant}$ have parallel normal direction.
(iii) The local holonomy group is at most two dimensional.

The method we use to prove this theorem is that of moving frames of E Cartán (for our notation see Plebański (1964), Boyer *et al* 1980)). We give the Cartán structure equations written in spinor form (summation convention is used)

$$\begin{aligned} &\mathrm{d}\theta^{A\dot{A}} + \omega^{A}{}_{B} \wedge \theta^{B\dot{A}} + \omega^{\dot{A}}{}_{\dot{B}} \wedge \theta^{A\dot{B}} = 0 \\ &\mathrm{d}\omega^{A}{}_{B} + \omega^{A}{}_{C} \wedge \omega^{C}{}_{B} = -C^{A}{}_{BCD}S^{CD} + \tfrac{1}{12}RS^{A}{}_{B} + C^{A}{}_{B\dot{C}\dot{D}}S^{\dot{C}\dot{D}} \end{aligned} \tag{2.7}$$

where $\theta^{A\dot{A}}$ is a spinor moving coframe and $\omega^{A}{}_{B}$ $(\omega^{\dot{A}}{}_{\dot{B}})$ is the connection 1-form with respect to this coframe. $C^{A}{}_{BCD}$ and $C^{A}{}_{B\dot{C}\dot{D}}$ are the spinor components of the Weyl tensor and traceless Ricci tensor, respectively. R is the scalar curvature, S^{AB} and $S^{\dot{A}\dot{B}}$ are 2-forms defined by

$$\begin{aligned} S^{AB} &= \tfrac{1}{2}\varepsilon_{\dot{A}\dot{B}}\theta^{A\dot{A}} \wedge \theta^{B\dot{B}} \\ S^{\dot{A}\dot{B}} &= \tfrac{1}{2}\varepsilon_{AB}\theta^{A\dot{A}} \wedge \theta^{B\dot{B}} \end{aligned} \tag{2.8}$$

where ε_{AB} $(\varepsilon_{\dot{A}\dot{B}})$ is antisymmetric and equal to 1 for $A = 1$, $B = 2$.

If we now reduce the spinor group $SL(2, C)$ to the subgroup which stabilises a principal null direction, say $\theta^{2\dot{2}}$ in coframe form, then the quantity $T = \omega^{2}{}_{1}$ is an invariant of the reduced principal bundle. With this in mind a convenient choice of coframe is

$$\begin{aligned} &2^{1/2}\theta^{2\dot{2}} = \frac{\phi^2}{A}\,\mathrm{d}x^2 \qquad && 2^{1/2}\theta^{1\dot{1}} = h\,\mathrm{d}x^2 + 2h_{2\alpha}\,\mathrm{d}x^{\alpha} \\ &\theta^{1} = \frac{1}{A}\,\mathrm{d}x^1 \qquad && \theta^{2} = \frac{\phi}{A}\varepsilon_{\alpha\beta}h^{\beta}\,\mathrm{d}x^{\alpha} \end{aligned} \tag{2.9}$$

where $h^{\alpha} = (a_2, b_2)$. It is easy to check that $\theta^{2\dot{2}}$ is indeed a principal null direction. Moreover, from (2.7), a straightforward computation gives

$$T = -2^{-3/2}(2A_{x_1} + \mathrm{i}\,\eta A\phi)\theta^{2\dot{2}} \tag{2.10}$$

where $\eta = a_2b_2' - b_2a_2'$. Now the 1-form $\theta^{2\dot{2}}$ defines a foliation of the space into three-dimensional (null) surfaces. Furthermore, using the $\mathrm{d}T$ equation of (2.7), we have the following theorem.

Theorem 2. In order that (M, g) admit a coordinate system of type D, it must have:

(i) a non-expanding, twistfree, shearfree, geodesic principal null direction, thus every leaf L_p of the foliation defined by $x^2 = \text{constant}$ is a totally geodesic surface,
(ii) an algebraically degenerate conformal curvature.

3. Proof of the main theorem

As a first step toward proving theorem 1, we prove a lemma which is of interest in its own right.

Lemma 3. Let (M, g) admit a coordinate system of type D (i.e. the Hamilton–Jacobi equation (2.1) is separable); then the Helmholtz equation (2.2) is separable in (M, g) if and only if there are constants α_i, $i=1,\ldots,12$, such that up to equivalence one of the following holds:

(1) $K_1'K_2' \neq 0 \qquad a_2b_2 \neq 0 \qquad \xi = a_2/b_2 \qquad e_1 = \alpha_1 f_1 + \alpha_2 d_1 + \alpha_3$

$$K_1 = \frac{\alpha_4 f_1 + \alpha_5 d_1 + \alpha_6}{\alpha_7 f_1 + \alpha_8 d_1 + \alpha_9} = \frac{\alpha_7 f_1 + \alpha_8 d_1 + \alpha_9}{\alpha_{10} f_1 + \alpha_{11} d_1 + \alpha_{12}}$$

$$(\xi^{-1} + \alpha_2\xi)(-\tfrac{1}{2}\alpha_{10}K_2^2 + 2\alpha_2 K_2 - \alpha_4) = (2 - \alpha_1\xi)(\tfrac{1}{2}\alpha_{11}K_2^2 - \alpha_4 K_2 + \tfrac{1}{2}\alpha_5)$$

$$f_2 = (\xi^{-1} + \alpha_2\xi)\frac{(-\tfrac{1}{2}\alpha_{12}K_2^2 + \alpha_9 K_2 - \tfrac{1}{2}\alpha_6)}{\alpha_{10}K_2^2 - 2\alpha_8 K_2 + \alpha_5}$$

(2) $K_1' = 0 \qquad \xi = a_2/b_2 \qquad b_2 \neq 0 \qquad \xi' \neq 0$

(*a*) $f_1' \neq 0 \qquad e_1 = \alpha_1 f_1 + \alpha_2 \qquad d_1 = \alpha_3 f_1 + \alpha_4$

$f_2 = \alpha_5 + \tfrac{1}{2}(\alpha_2 - \alpha_1\alpha_5)\xi + \tfrac{1}{2}(\alpha_4 - \alpha_3\alpha_5)\xi^{-1}$

(*b*) $f_1' = 0 \qquad e_1 = \alpha_1 d_1 + \alpha_2 \qquad d_1' \neq 0$

$f_2 = (\alpha_1\alpha_3 + \tfrac{1}{2}\alpha_2)\xi + \alpha_3\xi^{-1}$

(*c*) $f_1' = e_1' = d_1' = 0$

(3) $K_1' = 0 \qquad b_2 = \alpha_1 a_2$

(*a*) $f_2' = 0$

(*b*) $e_1 = 2\alpha_1 f_1 - \alpha_1^2 d_1 + \alpha_2$

(*c*) $\alpha_1 = 0$

(4) $K_2' = 0 \qquad K_1' \neq 0 \qquad \xi \equiv a_2/b_2 \qquad \xi' \neq 0$

(*a*) $f_1' \neq 0 \qquad e_1 = \alpha_1 f_1 + \alpha_2 \qquad d_1 = \alpha_3 f_1 + \alpha_4$

$f_2 = \alpha_5 + \tfrac{1}{2}(\alpha_2 - \alpha_1\alpha_5)\xi + \tfrac{1}{2}(\alpha_4 - \alpha_3\alpha_5)\xi^{-1}$

(*b*) $f_1' = 0 \qquad e_1 = \alpha_1 d_1 + \alpha_2 \qquad d_1' \neq 0$

$f_2 = \alpha_3\xi^{-1} + (\tfrac{1}{2}\alpha_2 + \alpha_1\alpha_3)\xi$

(*c*) $f_1' = e_1' = d_1' = 0$

(5) $K_2' = 0 \qquad K_1' \neq 0 \qquad b_2 = \alpha_1 a_2 \neq 0$

(*a*) $f_2' = 0$

(*b*) $e_1 = 2\alpha_1 f_1 - \alpha_1^2 d_1 + \alpha_2.$

Proof. As the proof involves some straightforward but rather tedious computation, we give the outline only. In Boyer *et al* (1978) it was shown that a necessary and sufficient condition for Helmholtz separability is

$$R_{12} = \tfrac{3}{4}\partial_{x^1x^2} \ln[(K_1 - K_2)^2/\psi] = 0.$$

This condition implies the existence of functions $\lambda_i(x^i)$, $i=1, 2$, such that $(K_1 -$

$K_2)^2/\lambda_1\lambda_2 = \psi$. If we define $\tilde{a}_2 = \lambda^{1/2} a_2$, $\tilde{b}_2 = \lambda^{1/2} b_2$, then we have

$$(K_1^2 - 2K_1K_2 + K_2^2)/\lambda_1 = 2\tilde{a}_2\tilde{b}_2(f_1+f_2) - \tilde{a}_2^2 e_1 - \tilde{b}_2^2 d_1.$$

Assuming $K_2' \neq 0$ define the operator $L = K_2'^{-1}\partial_{x^2}$. Then the integrability condition for the above equation is

$$L^3(2\tilde{a}_2\tilde{b}_2 f_1' - e_1'\tilde{a}_2^2 - d_1'\tilde{b}_2^2) = 0$$

and this equation has the general form

$$A_1A_2 - B_1B_2 - C_1C_2 = 0$$

where A_i, B_i, C_i are functions of x^i only. Assuming that none of these functions vanish this equation has two solutions: (i) A_2, B_2, C_2 are proportional and A_1, B_1, C_1 are linearly dependent and (ii) the same with 1 and 2 interchanged. Analysis of case (i) leads to coordinates of type (1), whereas case (ii) leads to $K_1' = 0$. The degenerate cases where one or more of the A_i, B_i, C_i vanish also leads to $K_1' = 0$. Analysing these cases leads to coordinates of type (2) and (3). Similarly, the case $K_2' = 0$ gives coordinates of type (4) and (5).

To proceed further we analyse the first two components of the second structure equations. This gives all except the component $C_{22\dot{2}\dot{2}}$ of the traceless Ricci tensor. We find that $C_{11\dot{1}\dot{1}}$ and $C_{11\dot{1}\dot{2}}$ are identically zero and

$$\begin{aligned}
C_{11\dot{2}\dot{2}} &= \frac{A^2}{2}\left(-\frac{A_{x^1x^1}}{A} + \frac{\phi_{x^1}}{\phi}\frac{A_{x^1}}{A} - \frac{\eta^2\phi^2}{4}\right)\\
C_{12\dot{1}\dot{2}} &= \frac{A^2}{4}\left(\frac{\phi_{x^1x^1}}{\phi} - \frac{A_{x^1x^1}}{A} - \frac{A_{x^1}\phi_{x^1}}{A\phi} + \frac{3\eta^2\phi^2}{4}\right)\\
C_{12\dot{2}\dot{2}} &= -(A^3/2\phi^3)[\tfrac{3}{2}(\ln\phi/A^2)_{x^1x^2} + \mathrm{i}(\tfrac{1}{2}\eta\phi + \eta H\phi^3 - \tfrac{1}{2}\eta\phi_{x^2}\\
&\quad - \eta\phi A_{x^2}/A - \tfrac{3}{2}\zeta_1\phi\phi_{x^1} + \zeta_1\phi^3(\ln A)_{x^1} - \tfrac{1}{2}(\zeta_1)_{x^1}\phi^3)]
\end{aligned} \tag{3.1}$$

where $\eta = a_2b_2' - b_2a_2'$, $H = h^\alpha(h_\alpha)_{x^2}$ and $\zeta_1 = (h_\alpha)_{x_1}\varepsilon^{\alpha\beta}h_\beta$. The scalar curvature can be written as (using $C_{12\dot{1}\dot{2}} = C_{11\dot{2}\dot{2}} = 0$)

$$R = 4[3(A_{x^1})^2 - 4(\ln\phi)_{x^1}A_{x^1}A - \tfrac{3}{8}\eta^2\phi^2A^2]. \tag{3.2}$$

The last component of the second structure equation is the messiest, so it will be convenient first to solve the equations that we have. In fact we shall prove lemma 4.

Lemma 4. Let (M, g) be an Einstein space which admits a coordinate system of type D; then it must be either type (3) or (5*b*) of lemma 3.

To prove this lemma it is convenient to convert this to an algebraic problem by using the modern theory of systems of partial differential equations (see, for example, Pommaret 1978), so we shall very briefly discuss this language. Given a local coordinate chart x^i on M we can consider k smooth functions $y^a(x^i)$ as the components of a local section of a fibre bundle $\mathscr{E}$ of fibre dimension r over M. Then one constructs another bundle $J_k(\mathscr{E})$—the bundle of k-jets over $\mathscr{E}$. A section of $J_k(\mathscr{E})$ (a k-jet of a section of $\mathscr{E}$) is just given by the functions $y^a(x^i)$ and all their derivatives up to order k. Thus a standard coordinate chart for $J_k(\mathscr{E})$ is $(x^i, y^a, y^a_{i_1}, y^a_{i_1i_2}, \ldots, y^a_{i_1,\ldots,i_k})$ where the $y^a_{i_1,\ldots,i_j}$ are symmetric in the lower indices. Notice that for $q > k$ there is a natural projection map $\pi^q_k: J_q(\mathscr{E}) \to J_k(\mathscr{E})$ sending a q-jet onto the corresponding k-jet. Now any system of partial differential equations (in general nonlinear) of order k is just a fibred submanifold $\mathscr{R}_k$ of $J_k(\mathscr{E})$. In terms of coordinates such a submanifold is just given by equations of the form

$$\Phi^\tau(x^i, y^a, y^a_i, \ldots, y^a_{i_1,\ldots,i_k}) = 0.$$

To investigate the integrability conditions for such equations one considers the prolongations of $\mathscr{R}_k$. That is, the first prolongation $\mathscr{R}_{k+1}$ of $\mathscr{R}_k$ is given by adding to $\mathscr{R}_k$ the equations obtained classically by taking the 'total derivative', namely

$$\frac{\partial \Phi^\tau}{\partial x^i} + \frac{\partial \Phi^\tau}{\partial y^a} y^a_i + \frac{\partial \Phi^\tau}{\partial y^a_\mu} y^a_{\mu+1_i} = 0$$

where we use the standard multi-index notation $\mu = (i_1, \ldots, i_j)$. The rth prolongation $\mathscr{R}_{k+r}$ is defined inductively. The (formal) integrability conditions for the system $\mathscr{R}_k$ are then given by the surjectivity of the maps $\pi^{k+r}_k : \mathscr{R}_{k+r} \to \mathscr{R}_k$ for all $r = 1, 2, \ldots$.

Proof of lemma 3.1. Since all functions only depend on (x^1, x^2) we have a two-dimensional base manifold with these as local coordinates. We introduce the coordinates (y^1, y^2, y^3) of $\mathscr{E}$ so that the corresponding section is $(y^1 = A, y^2 = \phi, y^3 = \eta)$. We consider the system $\mathscr{R}_2 \subset J_2(\mathscr{E})$ obtained from the equations $\eta_{x^1} = 0$, $R = 4\lambda_0$ (constant), $C_{11\dot{2}\dot{2}} = 0$, and $C_{12\dot{1}\dot{2}} = 0$ where we have used the fact that in an Einstein space the scalar curvature is constant. After rearrangement the system $\mathscr{R}_2$ can be written in our new notation as

$$\mathscr{R}_2 \begin{cases} y^3_1 = 0 \\ 3(y^1_1)^2 y^2 - 4y^1_1 y^2_1 y^1 - \frac{3}{8}(y^3)^2(y^2)^3(y^1)^2 - \lambda_0 y^2 = 0 \\ y^1_{11} y^2 - y^1_1 y^2_1 + \frac{1}{4}(y^3)^2(y^2)^3 y^1 = 0 \\ y^2_{11} y^1 - 2y^1_1 y^2_1 + (y^3)^2(y^2)^3 y^1 = 0. \end{cases}$$

We are interested in points of $\mathscr{R}_2$ with $y^1 y^2 \neq 0$. Instead of working with $\mathscr{R}_2$ directly, it will be convenient to consider $\mathscr{R}^{(1)}_1 \equiv \pi^2_1 \mathscr{R}_2$ given explicitly by the two first-order equations of $\mathscr{R}_2$. Every solution of $\mathscr{R}_2$ must be a solution of not only $\mathscr{R}^{(1)}_1$ but of all its prolongations $\mathscr{R}_{(1)+r}$, $r = 1, 2, \ldots$. We shall now make the assumption $y^1_1 y^2_1 \neq 0$ and show that $\mathscr{R}_2$ has no such solutions.

Now with the above assumption it is not difficult to see that $(\mathscr{R}^1_1)_{+1} \supset \mathscr{R}^{(1)}_2 = \pi^3_2 \mathscr{R}_3$ and so we consider $\mathscr{R}^{(2)}_1 = \pi^2_1 \mathscr{R}^{(1)}_2$ given explicitly by $\mathscr{R}^{(1)}_1$ plus

$$8y^1_1(y^2_1)^2 - \tfrac{7}{4}y^1_1(y^3)^2(y^2)^4 + \tfrac{1}{2}y^2_1(y^3)^2(y^2)^3 y^1 = 0.$$

We now continue this process to the higher prolongations, assuming without loss of generality by the above equation that $y^3 \neq 0$. We obtain $\mathscr{R}^{(4)}_1 = \pi^5_1 \mathscr{R}_5$ given by

$$\mathscr{R}^{(4)}_1 \begin{cases} y^3_1 = 0 \\ 3(y^1_1)^2 y^2 - 4y^1_1 y^2_1 y^1 - \frac{3}{8}(y^3)^2(y^2)^3(y^1)^2 - \lambda_0 y^2 = 0 \\ 8y^1_1(y^2_1)^2 - \frac{7}{4}y^1_1(y^3)^2(y^2)^4 + \frac{1}{2}y^2_1(y^3)^2(y^2)^3 y^1 = 0 \\ \frac{7}{2}(y^1_1)^2(y^2)^2 - (y^2_1)^2(y^1)^2 - \frac{45}{2}y^1_1 y^2_1 y^1 y^2 - \frac{1}{16}(y^3)^2(y^2)^2(y^1)^2 = 0 \\ 134(y^2_1)^2(y^1)^2 - 217(y^1_1)^2(y^2)^2 + 365y^1_1 y^2_1 y^1 y^2 = 0. \end{cases}$$

Every solution of $\mathscr{R}_2$ with $y^3 y_1^1 y_1^2 \neq 0$ must also be a solution of $\mathscr{R}_1^{(4)}$. We define $\alpha = y_1^1 y^2 / y^1 y_1^2$ and a straightforward computation shows that the last three equations of $\mathscr{R}_1^{(4)}$ imply

$$217\alpha^2 - 365\alpha - 134 = 0$$

$$49\alpha^3 + 329\alpha^2 - 86\alpha - 4 = 0.$$

It is not difficult to show that these polynomials have no common roots. We conclude that any solution of $\mathscr{R}_2$ must have $y_1^1 y_1^2 = 0$.

It is now an easy matter to show that any solution of $\mathscr{R}_2$ must be of one of the two forms

$$y_1^1 = y^3 = \lambda_0 = 0$$

$$y_1^2 = y^3 = 0 \qquad y_1^1 = \text{constant} \neq 0.$$

The first case is easily seen to be case (3) of lemma 3, while the second case using $R_{12} = 0$ gives $y_2^1 = 0$ and leads to case (5*b*) of lemma 3. This proves lemma 4.

To finish the proof of theorem 1 the aforementioned cases are analysed separately. Here we only give an outline. Analysing the final structure equation for case (5*b*) one sees that lemma 2 is applicable and this case is equivalent to a coordinate system with three ignorable coordinates. The space is Petrov type III with non-vanishing cosmological constant (Boyer *et al* 1979).

Cases (3*a*) and (3*c*) can be analysed jointly and this leads to metrics (1) and (2) of the theorem. For type (1) metrics the only non-vanishing curvature component is

$$C^1 = C_{2222} = -\frac{\alpha_1 A^4}{(x^1)^6} \tag{3.3}$$

and the Killing tensor for this case is

$$L_3 = \partial_{x^1x^1} + (\alpha_0 (x^1)^2 + \alpha_1 \ln x^1)\partial_{x^3x^3} - (x^1)^{-2}(\varepsilon\partial_{x^3} - \partial_{x^4})^2. \tag{3.4}$$

For type (2) metrics the curvature is

$$C^1 = \mp A^4 \tag{3.5}$$

and the Killing tensor is

$$-L_3 = \partial_{x^1x^1} \pm (x^1)^2 \partial_{x^3x^3} + [(\alpha_1 x^1 + \alpha_0)\partial_{x^3} - \partial_{x^4}]^2. \tag{3.6}$$

Case (3*b*) leads to metrics (3) and (4) of the theorem with

$$C^1 = -\frac{(\beta_1 A)^4}{\phi}\left(\frac{\phi''}{\phi} + \frac{\phi^2}{4} - \alpha_1 - \frac{3\mathrm{i}\phi'}{2}\right) \tag{3.7}$$

$$L_3 = \partial_{x^1x^1} + (\gamma_2 (x^1)^2 + \gamma_1 x^1)\partial_{x^3x^3} + x^1 \partial_{x^3x^4} \tag{3.8}$$

for type (3) metrics and

$$C^1 = \left(\frac{A}{\phi}\right)^4 \left(\alpha_1 - \frac{\phi''}{\phi}\right) \tag{3.9}$$

$$L_3 = \partial_{x^1x^1} + \alpha_1 (x^1)^2 \partial_{x^3x^3} \tag{3.10}$$

for metrics of type (4).

In all of the four cases, $T = 0$ and C^1 is the only non-vanishing curvature component and this gives the corollary.

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